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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

TECHNICAL MEMORANDUM 118

GENERAL PROBLEM OF THE AIRPLANE.

By Paul and Maurice Richard.

From "Premier Congrès International de la Navigation Aérienne," Paris, November, 1921, Vol. I.

1.7.1.3

July, 1922.

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FIRST PART.

I

The formulas hitherto employed for solving the general problem of the airplane are incomplete. They are, in fact, only the homogeneity ratios of flight equations and it is as independent variables that the quantities related to each other by the laws of the strength of materials figure in them. Everything concerning the mutual relations of flight and structure is therefore a closed domain for them. An important part of the problem is, by this very fact, abandoned to empiricism and arbitrary methods. Such are the discussions without logical results, often followed by unsuccessful experiments which take the place of the mathematically correct solution. Hence the uncertainty in regard to fundamental questions, such as: Thick wings or thin? What materials should be used in any given case? What results can we hope to obtain? etc. In reality, the aerodynamic problem is inseparable from the mechanical problem and both kinds of equations should together constitute the starting point for every airplane project. Consequently, the problem is perfectly defined and the solution, which could formerly be obtained only by groping and in an uncertain manner, is now obtained mathematically in all its general

* From "Premier Congrès International de la Navigation Aérienne," Paris, November, 1921, Vol. I, pp. 24-32.

applications. This is what we have tried to show in the present article. It will be seen, by the importance of the problems already solved, that if the new equations which figure in it are the necessary complement of the formulary of the engineer who is designing an airplane, they cannot be ignored by those who have to draw up the plan of aviation itself. They enable them to solve ordinary formulas without risk of falling into the impossible and to see clearly what programs to lay out and how to direct the efforts of investigators toward points most capable of bringing about the desired results.

II - KIND OF PROBLEMS CONSIDERED.

The problem of the airplane depends on a certain number of quantities, the variation of which is limited:

1. By their very nature. Example. - A speed or a carrying capacity may vary from 0 to infinity. The modulus of elasticity of existing substances varies from 0 to some definite value. The efficiency of present propellers varies from 0 to N %, etc.

2. By the system of airplane equations, which cannot be solved by every combination of values, separately possible, of the variables.

Such being the case, we can state two kinds of problems.

All the variables, less one, varying in their whole domain, it may happen that the variation of the latter is limited. For example, there is a certain speed which it is impossible to exceed in flight, a carrying capacity which can never be surpassed, etc.

Inversely, if a definite value is assigned to one of the variables, it may happen that the problem has no solution, if the other variables remain in their domain. The study of the conditions of reality will therefore lead to the determination of what progress it would be possible to realize in such or such branch of the art in order that a given performance may become possible.

Along with these general problems, there are particular problems of no less practical interest. If the field of one or more of the variables is arbitrarily limited, what is the correlative limitation of the field of another variable? For example, what is the maximum speed obtainable with a given carrying capacity, varying within given limits? What is the longest radius of action obtainable with wooden airplanes, etc.? The solution of these last problems determines the materials to be employed in any given case.

By way of illustration, we will enumerate a few problems which we have studied.

The wing section and parasite resistance being given, to determine the maximum characteristics of an airplane.

I. <u>Military or "record" airplanes.</u>

1. Speed:

    a) Without conditions;
    b) With given carrying capacity.

2. Altitude:

    a) Without condition;
    b) With given carrying capacity.

II. Commercial airplanes.

3. Carrying capacity:

   a) Without condition;
   b) With a given speed.

4. Radius of action:

   a) Without condition;
   b) With a given speed.

III - GENERAL EQUATIONS.

Weight of airplane.- The total weight W of an airplane comprising:

1. The useful load with its accompanying "dead weight" which is proportional to it, or u.

2. Weight of engine with all that is required for its support and operation. This term is proportional to the engine power T for a given number of hours of flight. It can be expressed in either of the following forms:

$$qT, \qquad rT + s, \qquad (r + cn)T,$$

s being the total quantity of gasoline carried, c the hourly consumption and n the number of hours of flight.

3. The weight aW (proportional to the total weight W) of certain parts, such as the hull of a seaplane, the landing gear of an airplane, etc.

4. Weight of the glider, which may be expressed in the form $W \frac{AS^{\frac{1}{2}}}{1+AS^{\frac{1}{2}}}$, W being total weight of airplane, S its area and A

a coefficient depending on the kind of construction. This formula which is absolutely general for the cell, may be written

$$p_o = A\ (W - p_o)\ \lambda .$$

In this form, it expresses the fact that the weight p_o of the cell is proportional to the load $W - p_o$ for which it is calculated and to the ratio λ of the linear dimensions. Now, in homothetic girders, homologous sections remain the same, the lever arms of stresses in the girders being multiplied by the same number λ as the lever arm of the loads $(W - p_o)$.

Elements calculated to the point of collapse show no derogation of this law, since the section required by the compression is maintained by modifying its form so as to give it the necessary strength. The law still holds true for continuous girders with full web (or cross-section), the calculation of which is derived from the discontinuous case by summation or integration; and of which the multiple supports only have the effect of introducing linear combinations of moments. All these propositions have been verified in detail.

5. The weight of the elevator p_q varies according to the same law. Anyway, since the weight p_q is either added or subtracted according to the direction of the stress, it is more exact to write $p_q = BS^{\frac{1}{2}}kW$. Since the part of the fuselage which supports the tail is expressed in this same form, it may be included in the coefficient B.

6. Weight of covering, which is proportional to S. By summing up, we obtain

$$W = W\left[\frac{AS^{\frac{1}{2}}}{1 + AS^{\frac{1}{2}}} + BkS^{\frac{1}{2}} + a\right] + CS\,qT + u.$$

Experience has shown that (save for exceptional cases, for example, that of a very heavy metal covering) the terms $S^{\frac{1}{2}}$ and S may be included in the principal term, with the aid of a suitable modification of the coefficient A. The resulting error may reach 3.5% for a surface variation of 5 square meters in 10000. The final formula is therefore

$$W = W\left(\frac{AS^{\frac{1}{2}}}{1 + AS^{\frac{1}{2}}} + a\right) qT + u.$$

IV - POLARS OF THE WINGS.

It is found that all the important wing polars may be represented with an approximation superior to that of the experiments which served to establish them by a formula of the form

$$K_x = \alpha + \mu\,(K_y - \beta)^2$$

α, μ and β being constant coefficients varying from one polar to another.

Fundamental formulas.- The following are the fundamental formulas employed:

(1) $$W = W\left(\frac{AS^{\frac{1}{2}}}{1 + AS^{\frac{1}{2}}} + a\right) + qT + u$$

(2) $$\Omega = \frac{VS^{\frac{1}{3}}}{T^{\frac{1}{3}}}$$

(3) $$W = d^{\frac{1}{3}}\,T^{\frac{2}{3}}\,S^{\frac{1}{3}}$$

(4) $$\left(\mu\,\beta^2 + \frac{\sigma}{S} + \alpha\right) W^2\,\Omega^2 - 2\,\mu\,d^{\frac{1}{3}}\beta\,W\,\Omega^2 - \rho\,75\,t\,W^3\Omega + \mu d^{\frac{2}{3}} = 0.$$

The first is the equation of weight.

The second defines, in terms of the speed V, the surface area S and the power T, a variable Ω which it is convenient to consider.

The third is the usual equation for the ceiling.

The fourth (in which $\frac{\sigma}{S}$ is the coefficient of fineness, W the ratio $\frac{\delta}{\delta_0}$ of the densities, t the ratio $\frac{T}{T_0}$ of the absolute temperatures and ρ the average propeller efficiency) is only the equation of the polar already mentioned expressed in terms of the variable Ω.

SECOND PART.

I - MAXIMUM SPEED.

Restricted Problem.

The ceiling is given, i.e. the value of d and the useful load u. Under these conditions, equation (4) gives the value of Ω immediately.

By eliminating W and T in equations (1), (2), and (3), we obtain

$$f(z,v) = d^{\frac{1}{3}} v^2 \left(\frac{1}{z} - a\right) - q v^3 - \frac{u A^2}{(z-1)^2} = 0$$

in which $v = \frac{V}{\Omega}$, $z = 1 + AS^{\frac{1}{2}}$

which by the elimination of V between it and its derivative in z brings us to the equation

$$2\tau - Aq\sqrt{8\frac{u}{d}}\,v^3\gamma - \frac{1}{\gamma^2} = 0$$

with $\qquad \tau = 1 - az \qquad v = \sqrt{\frac{z}{z-1}} \qquad \gamma = \sqrt{z}$

from which is derived the value of the surface area necessary to obtain the maximum speed. The equation $f(z_1 v) = 0$ then gives this speed and the other variables are derived from it: T from equation (2) and W from equation (3). The characteristics of an airplane of maximum speed are thus perfectly determined.

General Problem.

The discussion of the preceding equations shows that the maximum speed is a decreasing function of u and of q. Hence for the maximum speed, an airplane must carry only its pilot, be equipped with the lightest possible engine per HP and carry only sufficient fuel for accomplishing the required performance.

There remains to be determined V_{max} in terms of Ω. The equations

$$d^{\frac{2}{3}}\frac{V^2}{\Omega^2}\left(\frac{1}{z} - a\right) + q\frac{V^3}{\Omega^3} - \frac{uA^2}{(z-1)^2} = 0$$

and

$$\frac{d^{\frac{1}{3}}}{\Omega^2} = \frac{1}{\sqrt{\mu}}\left[\sqrt{\frac{\rho\ 75t}{\Omega^3} - \frac{\sigma}{S} - \alpha} + \beta\right]$$

give, by the elimination of $\frac{d^{\frac{1}{3}}}{\Omega^2}$

$$\frac{V^2}{\sqrt{\mu}}\left(\frac{1}{z} - a\right)\left[\beta + \sqrt{\frac{\rho\ 75t}{\Omega^3} - \frac{\sigma}{S} - \alpha}\right] + \frac{qV^3}{\Omega^3} - \frac{uA^2}{(z-1)^2} = 0$$

By eliminating Ω between this equation and the derived equation in Ω, we obtain

$$\frac{V^2}{\sqrt{\mu}}\left(\frac{1}{z} - a\right)\left(\beta + \frac{k}{V}\right) - \frac{qV^3}{\rho\, 75t}\left(\frac{k^2}{V^2} + \frac{\sigma}{S} + \alpha\right) - \frac{uA^2}{(z - 1)^2} = 0$$

with

$$k = \frac{\rho\, 75t}{2\sqrt{\mu q}}\left(\frac{1}{z} - \alpha\right)$$

from which is obtained the value of V which gives the general solution of the problem.

II - MAXIMUM ALTITUDE.

General Problem.

Formulas (1) and (2), by the elimination of W, give

$$d^{\frac{1}{3}}\, T^{\frac{2}{3}}\, S^{\frac{1}{3}}\left(a - \frac{1}{1 + AS^{\frac{1}{2}}}\right) + qT + u = 0$$

in which the variables are separated. The conditions for the maximum of d are only those for the minimum of $\frac{qT + u}{T^{\frac{2}{3}}}$ and for the maximum of $S^{\frac{1}{3}}\left(a - \frac{1}{1 + AS^{\frac{1}{2}}}\right)$ that is, $T = \frac{2u}{q}$ and

$2aA^2S + A(1 + 4a)S^{\frac{1}{2}} + 2(a - 1) = 0$ whence the solution of the problem. The minimum $\frac{3}{2^{\frac{2}{3}}}\, u^{\frac{1}{3}}\, q^{\frac{2}{3}}$ of the function $\frac{qT + u}{T^{\frac{2}{3}}}$ being lower just in proportion as u and q are smaller, we take for u the weight of the pilot and a value of q corresponding to the lightest engine and the exact amount of gasoline required for accomplishing the performance.

Restricted Problem.

(Maximum altitude for a given speed.)

Formula (4) gives $d^{\frac{1}{3}}$ in terms of Ω.

The derivative $\frac{d d^{\frac{1}{3}}}{d\Omega}$, made equal to 0, gives

$$\frac{1}{4\mu} 75^2 \rho^2 t^2 x^2 - 150 \left(\rho t \sqrt{\beta^2 - \frac{1}{\mu} \alpha + \frac{\sigma}{S}}\right) x + \left(\alpha + \frac{\sigma}{S}\right)\left(\frac{4}{\mu}\alpha + \frac{\sigma}{S} + 4 \beta^2\right) = 0$$

with

$$x = \frac{1}{\Omega^3}$$

The value of Ω gives that of d by formula (4) and the problem is then solved without difficulty.

III - MAXIMUM USEFUL LOAD.

General Problem.

The derivation of equation (1), which combines the weights (or loads), gives for $\frac{du}{dS} = 0$, the equation

$$2 a A^2 S + (4 a A + A) S^{\frac{1}{2}} + 2(a - 1) = 0$$

and for $\frac{du}{dT} = 0$ $\qquad \frac{qT}{u} = 2,$

which are the equations intervening in the problem of the absolute maximum ceiling.

Restricted Problem.

(Maximum useful load for a given speed.)

If d is given such a value that the airplane can fly, equation (4) immediately gives the value of Ω. This being the case, equation (1), in which W has been eliminated by the aid of formula (3) and T with the aid of formula (2), will assume the following form

$$\frac{u}{d^{\frac{1}{3}} v^{2}} = f(S) = \left(\frac{1}{z} - c\right) S$$

with

$$v = \frac{V}{\Omega} \qquad z = 1 + AS^{\frac{1}{2}} \qquad c = a + \frac{qv}{d^{\frac{1}{3}}}$$

The annulment of the derivative f'_S gives the equation

$$2\,c\,z^{2} - z - 1 = 0$$

whence

$$z = \frac{1 + \sqrt{1 + 8c}}{4c}$$

The determination of the other characteristics than presents no difficulty.

IV - MAXIMUM RADIUS OF ACTION.

The differential dL of the radius of action is given by the formula

$$dL = \frac{K}{B}\ \frac{dW}{W}$$

B being the fineness. If the latter is constant, which corresponds to the absolute maximum radius of action for flight at the angle of minimum power, the integration leads without difficulty to the classic form

$$L = \frac{K}{cB} \log \frac{1}{1 + \frac{W_1}{W_0}}$$

If, on the contrary, it is desired to find the maximum radius of action at the maximum speed, the fineness becomes variable and the radius of action is obtained as follows:

We have:

$$L = \int_{t_0}^{t_1} V\,d\,t = \frac{1}{c}\int_{W_1}^{W_0} V\,d\,W$$

c being the total consumption in kg/sec. Furthermore, the elimination of $d^{\frac{1}{3}}$ and Ω in formulas (2), (3) and (4) leads to the equation

$$\left(\alpha + \frac{\sigma}{S} + \mu\,\beta^2\right)V^4 - \frac{2\,\mu\,\beta}{S}\,W\cdot V^2 - \frac{\rho\,75T}{S}\,V + \frac{W^2}{S^2} = 0$$

an equation of the second degree in which renders it possible to calculate W and dW in terms of V and dV. The expression for radius of action then takes the form

$$L = \frac{S}{c}\int_{V}^{V_0}\left[2\beta V^2 + \frac{1}{2}\sqrt{\frac{\rho 75V}{\mu S} - \frac{\alpha+\frac{\sigma}{S}}{\mu}V^4} - \frac{3}{2}\,\frac{\left(\alpha+\frac{\sigma}{S}\right)V^4}{\mu}\cdot\frac{1}{\sqrt{\frac{\rho\,75V}{\mu S} - \frac{\alpha+\frac{\sigma}{S}}{\mu}V^4}}\right]dV$$

which, after integration, becomes

$$L = \frac{2}{3}\,\xi\,\frac{T}{c}\left[\frac{\beta}{\xi}\Omega^3 - \frac{1}{\xi}\Omega^3\sqrt{\frac{\xi}{\Omega^3} - \psi} - \operatorname{arctg}\frac{\sqrt{\frac{\xi}{\Omega^3} - \psi}}{\sqrt{\psi}}\right]$$

to be taken between the limits Ω_1 and Ω_0 and where

$$\varphi = \frac{\rho\,75T}{\mu^3} \qquad \psi = \frac{\alpha + \frac{\sigma}{S}}{\mu} \qquad \xi = \frac{S\varphi}{T}$$

General Problem.

One flies at the angle of least power.

The term qT being put in the form $rT + s$, equation (1) gives

$$\frac{s}{W} = \frac{1}{1 + AS} - a - \frac{u + rT}{W}.$$

By derivation and by annulment of the two derivatives $\frac{d\frac{s}{W}}{dT}$ and $\frac{d\frac{s}{W}}{dS}$, we obtain

$$T = \frac{2u}{r} \qquad \text{and} \qquad 2\,\psi\,A^2x^6 - 3\,A\,x^5 + 4\,A\,\psi\,x^3 + 2\,\psi = 0$$

in which

$$\psi = \frac{u + rT}{d^{\frac{1}{3}}\,T^{\frac{2}{3}}} \qquad r = S^{\frac{1}{6}}$$

These equations solve the problem.

Maximum radius of action for flight at a given speed and for a given "unballasting."

$TF(\Omega)$ is the expression for the radius of action.

Now, Ω depends only on the polar and d, which is itself a function of W. If W becomes ηW, d becomes $\eta^3 d$. If therefore the value of the expression $\frac{W-s}{W} = \eta$ is kn wn, Ω_0 and Ω_1 are also known. It follows that the result of the integration, the polar being given, depends only on η and, for a given η, the maximum radius of action η' will be given by the maximum of T.

We arrive at the equation

$$2\eta'' A^{7} S + (1 + 4\eta'')AS^{\frac{1}{2}} - 2(1 - \eta') = 0$$

with

$$a + 1 - \eta' = \eta''$$

A table of the maxima for different values of $\frac{S}{W}$ has been prepared, which gives the value of the absolute maximum.

V MAXIMUM SPEED WITH $\frac{\sigma}{S} = a + \frac{\Omega}{S} T$.

As an example of a case in which it may be of interest to throw off the general restrictions imposed at the beginning, we have added this problem, which relates to airplanes of small wing surface, in which the largest cross-section of the engine cannot be diminished so as to maintain a constant $\frac{\sigma}{S}$.

Equation (4) of the polar is converted into

$$f(\Omega) = a\Omega^{3} - \Omega V^{3} + t\,\rho\,75 = \mu \cdot \left(\frac{d^{\frac{1}{3}}}{W}\frac{1}{\Omega^{\frac{1}{2}}}\beta\Omega^{\frac{3}{3}}\right)^{2} + d\Omega$$

The annulment of the derivative $f'(\Omega)$ gives

$$3(\alpha - a - \beta^{2}\mu)\,\Omega^{4} + \frac{2\beta\mu d^{\frac{1}{3}}}{W}\,\Omega^{2} \quad \frac{\mu d^{\frac{1}{3}}}{W} = 0$$

whence the value of Ω and of $V\Omega$.

If the condition of reality

$$\frac{(z - 1)^{2}\;(1 - az)^{3}}{z^{3}} = \frac{27}{4}\,u\,\frac{A^{2}\,q^{2}}{d}$$

is fulfilled, the equation

$$d^{\frac{1}{3}} v^2 \left(\frac{1}{z} - a\right) - q\, v^3 - \frac{u\, A^2}{(z - 1)} = 0$$

gives z in terms of v and the problem is solved without difficulty.

CONCLUSION.

The results are presented in the form of tables with multiple entries which are divided into a certain number of tables with double entry.

Lack of space prevents us from giving examples and insisting on the numerous analytical and nomographic methods, which have made it possible to establish them in the simplest manner. Taken together, they constitute the general solution of the problem of aviation.